# HW 6 Ch. 3 #20, 21, 23

(20) Suppose $\{p_n\}$ is a Cauchy sequence in a metric space $X$, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to $p$.

pf/ ① $p_{n_i} \longrightarrow p \Rightarrow \forall \varepsilon > 0 \ \exists i \in \mathbb{N}$ s.t. $\forall j > i$
$d(p_{n_j}, p) < \varepsilon$. (Let $N_i \in \mathbb{N}$ be the largest number s.t. $n_j > N_i \ \forall j > i$.)

② $\{p_n\}$ Cauchy $\Rightarrow \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall m, n > N$
$d(p_n, p_m) < \varepsilon$.

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Let $\varepsilon > 0$, $i$ & $N_i$ be as in ①, and $N$ be as in ②. Define $M = \max\{N, N_i\}$ and $J \in \mathbb{N}$ s.t. $n_j > M \ \forall j > J$. Then for $n > M$ and $j > J$

$$d(p_n, p) \leq d(p_n, p_{n_j}) + d(p_{n_j}, p) < \varepsilon + \varepsilon = 2\varepsilon$$

Let $\varepsilon \to 0 \Rightarrow p_n \longrightarrow p$.

☐ Epic.

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space $X$, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

pf/ From each $E_n$ choose a point $x_n$. Then since $\operatorname{diam}(E_n) \to 0$, $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete $x_n \to x \in X$ and moreover, since each $E_n$ is closed it is complete and hence $x \in E_n$ $\forall n$ (since infinitely many terms of the sequence are in each $E_n$).

$\Rightarrow x \in \bigcap_1^\infty E_n = \mathcal{E}$.

Suppose $x, y \in \mathcal{E}$, $x \neq y$. Then $\operatorname{diam}(E_n) \geq d(x,y) > 0$ $\forall n$ which contradicts $\operatorname{diam}(E_n) \to 0$.

$\therefore$ $|\mathcal{E}| = 1$

$\square$ dpk

23 Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space $X$. Show that the sequence $\{d(p_n, q_n)\}$ converges.

pf/ ① $d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$

$\Rightarrow d(p_n, q_n) - d(p_m, q_m) \le d(p_n, p_m) + d(q_m, q_n)$

② $d(p_m, q_m) \le d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m)$

$\Rightarrow d(p_m, q_m) - d(p_n, q_n) \le d(p_m, p_n) + d(q_n, q_m)$

①+② $\Rightarrow -(d(p_m, p_n) + d(q_n, q_m)) \le d(p_n, q_n) - d(p_m, q_m) \le d(p_m, p_n) + d(q_n, q_m)$

$\Rightarrow |d(p_n, q_n) - d(p_m, q_m)| \le d(p_m, p_n) + d(q_n, q_m)$

Let $\varepsilon > 0$.

$\{p_n\}$ Cauchy $\Rightarrow \exists N_1 \in \mathbb{N}$ s.t. $\forall n, m > N_1$; $d(p_n, p_m) < \varepsilon$

$\{q_n\}$ Cauchy $\Rightarrow \exists N_2 \in \mathbb{N}$ s.t. $\forall n, m > N_2$; $d(q_n, q_m) < \varepsilon$.

Let $M = \max\{N_1, N_2\}$, then $\forall n, m > M$

$|d(p_n, q_n) - d(p_m, q_m)| \le d(p_m, p_n) + d(q_n, q_m) < \varepsilon + \varepsilon = 2\varepsilon$

$\Rightarrow \{d(p_n, q_n)\}$ is Cauchy in $\mathbb{R}$ $\Rightarrow$ it converges since $\mathbb{R}$ is complete